

The Density Matrix Renormalization Group, Quantum Groups and Conformal Field Theory

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Abstract

We present an overview of the Density Matrix Renormalization Group and its connections to Quantum Groups, Matrix Products and Conformal Field Theory. We emphasize some common formal structures in all these theories. We also propose two-dimensional extensions of the variational matrix product ansatzs.

I. INTRODUCTION

The Density Matrix Renormalization Group (DMRG) is a powerful numerical Real Space RG method introduced in 1992 by S.R. White which can be applied to a large variety of Quantum Lattice Hamiltonians defined in 1d, quasi 1d (ladders) and large clusters [1]. The DMRG was originally proposed in the domain of Condensed Matter Physics, where it has already become a standard method specially for 1d systems, but its range of applicability has been extended also to Statistical Mechanics, polymers, Chemical Physics, etc... (for a review see [2]) and one may expect that it will be applied in the near future to other domains of Physics as Quantum Field Theory, Nuclear Physics, etc.

There are by now excellent reviews on the DMRG [3] and other related real space RG methods [4], [5] so it is not the purpose of the present contribution to duplicate material already present in the literature. Instead we shall try to give an overview of the DMRG

method in order to explore its relationships with Group Theory and its quantum deformation, Conformal Field Theory (CFT) and the Matrix Product Method. The relation between the DMRG and Quantum Groups was suggested in [6] by studying the RG method of q -group invariant Hamiltonians. We also review the variational approach of the DMRG in terms of the so called matrix product (MP) ansatz introduced by Östlund and Rommer [7] and suggest a 2D versions of it which may lead to a 2D formulation of the DMRG.

REAL SPACE RG METHODS: GENERALITIES

Let us suppose we have a discrete system with N sites and that at each site there are two possible states, say spin up and down for a spin system, or occupied and unoccupied for a spinless fermion. The dimension of the Hilbert space will grow as 2^N , which makes very hard the study of the large N limit, unless some special trick is used. The RG method provides a general systematic approach to handle problems with a large number of degrees of freedom on the basic assumption that only a small number of states is needed in order to describe the long distance physics. How to choose the **most representative** degrees of freedom out of a myriad of states is the central issue of the RG method. This can be done in several manners. We shall introduce below some of them and establish their comparison.

The DMRG method was originally formulated as a real space RG method although it admits also a momentum space formulation [8]. We next introduce the basic concepts common to any real space RG method and later on we confine ourselves to the DMRG.

The real space RG consist essentially of 3 steps: i) blocking, ii) truncation and iii) iteration. First one divides the system into blocks, then one finds an effective description of these blocks in terms of intra-block and inter-block interactions and finally one iterates the algorithm. One can distinguish between 3 types of blockings Kadanof-blocking, Wilsonian-blocking and DMRG-blocking.

Kadanof Blocking

We shall first consider the case of a linear chain with N sites. In the first step of the Kadanof blocking one divides the system into blocks of 2 sites, thus for a $N = 16$ chain one gets,

$$(\bullet \bullet) (\bullet \bullet) (\bullet \bullet) (\bullet \bullet) (\bullet \bullet) (\bullet \bullet) (\bullet \bullet) (\bullet \bullet) \quad (1)$$

If every site describes two states then the block $(\bullet \bullet)$ describes 4 states. Eq.(1) is nothing but a change from a 1-site basis to a 2-sites basis and hence $\bullet \bullet$ is entirely equivalent to $(\bullet \bullet)$. The goal of eq.(1) is to prepare the road to perform the first RG truncation. Indeed out of the 4 states we may already want to keep a smaller number, say 3, 2 or even 1. We can represent symbolically this operation as follows,

$$(\bullet \bullet) \rightarrow (\bullet \bullet)' \text{ (truncation)} \quad (2)$$

From a formal point of view the blocking operation is captured by putting parenthesis (...) around the sites subjected to the blocking while the truncation operation is represented by ' acting on the corresponding block. Combining eqs. (1) and (2) the chain with N sites become after the first blocking and truncation,

$$(\bullet \bullet)' (\bullet \bullet)' (\bullet \bullet)' (\bullet \bullet)' (\bullet \bullet)' (\bullet \bullet)' (\bullet \bullet)' (\bullet \bullet)' \quad (3)$$

If only one state is kept in (3) then the RG process ends up since there is already a single state to represent the ground state of the system. A dimerized spin chain is a typical example of this type of states, where $(\bullet \bullet)'$ is given by the singlet formed by two spin 1/2 of the chain.

In general however one keeps more than one state per block $(\bullet \bullet)'$ and so one can continue the RG method choosing $(\bullet \bullet)'$ as a the new site \bullet' , i.e.

$$(\bullet \bullet)' \rightarrow \bullet' \quad (4)$$

$$((\bullet \bullet)' (\bullet \bullet)')' = (((\bullet \bullet)' \bullet)' \bullet)' \quad (9)$$

Eq.(8) is reminiscent of the associativity of the tensor product of representations in group theory and more precisely in quantum group theory (see below). This equation certainly holds if there is no truncation of degrees of freedom, i.e. $(B_1 B_2)' = (B_1 B_2)$ in which case it amounts to the equivalence between different basis. In group theory the relation between different basis is given by $6 - j$ symbols.

Quantum groups are q -deformations of classical groups ($q = 1$ in this notation) where some of the commutator and addition rules are deformed (for a review see [10]). The representation theory of quantum groups, when the deformation parameter q is generic is analogue to that of classical groups. However when q is a root of unit things change completely. First of all there are a finite number of regular irreps and the tensor product of them is also truncated while keeping the associativity condition (8).

The existence of an associative truncated tensor is a common feature of the DMRG, quantum groups and Conformal Field Theory (CFT) (more on this point below).

DMRG Blocking

There are two DMRG algorithms to study open chains. The infinite system algorithm uses the superblock $B_\ell \bullet \bullet B_\ell^R$ to grow the chain from both sides according to the Wilsonian scheme [1]. The block $B_\ell \bullet$ is then truncated to a new block $B'_{\ell+1}$. In this manner the size of the system grows indefinitely until one reaches a fixed point beyond which the numerical results reproduce themselves. This method is very good in computing bulk properties of the system like the ground state energy density. In many cases however it is more convenient to study finite size systems whose large distance properties can later on be obtained through finite size scaling techniques. This is notably the case of gapless systems.

The DMRG algorithm used in these cases is called the finite system method and it is extremely accurate. The first steps of this method uses the infinite system algorithm to grow both sides of the chain independently until the left and right blocks are a half the size

of the chain. The chain with N (even) sites is then obtained by joining a left block $B_{N/2}$ and a right block $B_{N/2}^R$ as follows (the superindex R in $B_{N/2}^R$ indicates that it can be obtained from the reflection of a right block),

$$((((((((((\bullet \bullet)' \bullet)' \bullet)' \bullet)' \bullet)' \bullet)' (\bullet (\bullet (\bullet (\bullet (\bullet (\bullet \bullet)')')')')')')')')')' \quad (10)$$

The superblock (10) is used in the DMRG to enlarge the left block from $B_{N/2}$ to $B_{N/2+1}$ while the right block is reduced from $B_{N/2}^R$ to $B_{N/2-1}^R$, in which case we get

$$((((((((((\bullet \bullet)' \bullet)' \bullet)' \bullet)' \bullet)' \bullet)' \bullet)' \bullet)' (\bullet (\bullet (\bullet (\bullet (\bullet (\bullet \bullet)')')')')')')')' \quad (11)$$

If the associativity eq.(8) would hold then the blockings (10) and (11) would give the same result for the GS of the whole chain but of course this is not the case. The next step is to again enlarge the left block at the expenses of the right one until one reaches the right hand side. There one reverses the trend and grows the right blocks at the expenses of the left ones. After several sweeps of this back-and-forth algorithm the GS energy and GS wave function converge to a fixed values which are independent of the size of left and right blocks.

In this moment the splitting of the chain into left and right blocks is independent of their size so that the associativity constraint (8) is effectively fulfilled.

The analysis performed so far is rather formal but helps to abstract the blocking procedure which is common to all the real space RG methods. As a by-product we have shown that the blocking and the iteration procedures have to be considered as combined strategies to achieve the same goal which is to reduce the whole system to a single effective site. From a formal point of view blocking is like tensoring representations. In this sense the RG steps can be seen as “putting parenthesis” around the blocks. An exact RG method would be the one for which the final result would be independent on the way the parenthesis are put on. These lead us to the associativity constraint (8), whose fulfillment is the actual goal of any exact RG method.

THE STANDARD RG ALGORITHM

In all the real space RG methods there is an algorithm to truncate the collection of two blocks $B_1 B_2$ down to a new effective block B'_{12} where B_1 or B_2 may stand also for a single site.

The standard RG algorithm consist of the following steps 1) diagonalization of the Hamiltonian $H_{B_1 B_2}$ for the combined block $B_1 B_2$, 2) truncation to the lowest energy states of $H_{B_1 B_2}$ and 3) change of basis to the new states kept and renormalization of the old Hamiltonian.

This method leads to a lot of problems whose origin was first pointed out in reference [11] following a suggestion by Wilson. Studying the very simple problem of a particle in a box the authors of reference [11] interpreted the bad performance of the standard RG as been due to an incorrect treatment of the boundary conditions applied on a block by its neighbours. In other words the truncation $B_1 B_2 \rightarrow (B_1 B_2)'$ has to take into account the presence of say a third block B_3 in contact with the former ones. The key idea is to consider a superblock $B_1 B_2 B_3$ where the effect of B_3 into the other two blocks can be properly considered. An alternative RG-solution to the particle-in-a-box problem, which also takes into account the effect of boundary conditions has been given in [12].

THE DMRG ALGORITHM

Let us choose a superblock made out of three blocks $B_1 B_2 B_3$. The middle block is taken to be a single site or two sites $B_2 = \bullet$ or $\bullet \bullet$. Then one constructs the Hamiltonian $H_{B_1 B_2 B_3}$ describing the dynamics of the superblock and finds out a given state called the target state, which is usually the ground state of the superblock which can be written as

$$|\psi\rangle = \sum_{i_1, i_2, i_3} \psi_{i_1 i_2 i_3} |i_1, i_2, i_3\rangle \quad (12)$$

where i_1, \dots run from 1 up to m_1, \dots . The superblock can be regarded either as $((B_1 B_2) B_3)$ or as $(B_1 (B_2 B_3))$. Correspondingly the target wave function can be written in two different manners,

$$\psi_{i_1 i_2 i_3} = \sum_{\alpha} U_{i_1 i_2, \alpha} D_{\alpha}^{(12)3} V_{\alpha, i_3} \quad (13)$$

$$\psi_{i_1 i_2 i_3} = \sum_{\beta} U_{i_1, \beta} D_{\beta}^{1(23)} V_{\beta, i_2 i_3}$$

where U and V are matrices which “diagonalize” the wavefunction and satisfy the orthogonality conditions,

$$\sum_{i_1 i_2} U_{i_1 i_2, \alpha}^* U_{i_1 i_2, \alpha'} = \delta_{\alpha, \alpha'} \quad (14)$$

$$\sum_{i_3} V_{\alpha, i_3}^* V_{\alpha', i_3} = \delta_{\alpha, \alpha'} \quad (15)$$

We have used in (13) the singular value decomposition (SVD) of a matrix [1]. $D_{\alpha}^{(12)3}$ and $D_{\beta}^{1(23)}$ are the singular values of ψ regarded as a $(m_1 m_2) \times m_3$ or as a $m_1(m_2 \times m_3)$ matrix. Eqs.(13) are the clue of the DMRG method. Let us imagine for a moment that $D_{\alpha}^{(12)3}$ and $D_{\beta}^{1(23)}$ are zero for certain values of α and β . In this case it is clear that we can truncate the states of $B_1 B_2$ (resp. $B_2 B_3$) down to a smaller set of states α (resp. β) for which $D_{\alpha}^{(12)3}$ (resp. $D_{\beta}^{1(23)}$) is non zero without losing any information in order to reconstruct the target state ψ . Rather than performing the SVD of ψ it is more convenient to define the density matrices for the subsystems (12) and (23) inside the whole system (123) [1],

$$\rho_{i_1 i_2, i'_1 i'_2}^{(12)} = \sum_{i_3} \psi_{i_1 i_2 i_3}^* \psi_{i'_1 i'_2 i_3} \quad (16)$$

$$\rho_{i_2 i_3, i'_2 i'_3}^{(23)} = \sum_{i_1} \psi_{i_1 i_2 i_3}^* \psi_{i_1 i'_2 i'_3}$$

Now using (13), (15) we get,

$$\begin{aligned} \rho_{i_1 i_2, i'_1 i'_2}^{(12)} &= \sum_{\alpha} U_{i_1 i_2, \alpha}^* \left(D_{\alpha}^{(12)3} \right)^2 U_{i'_1 i'_2, \alpha} \\ \rho_{i_2 i_3, i'_2 i'_3}^{(23)} &= \sum_{\beta} V_{\beta, i_2 i_3}^* \left(D_{\beta}^{1(23)} \right)^2 V_{\beta, i'_2 i'_3} \end{aligned} \quad (17)$$

Eqs.(17) means that $w_{\alpha}^{(12)} = (D_{\alpha}^{(12)3})^2$ are the eigenvalues of the density matrix $\rho^{(12)}$ while U is the unitary matrix which diagonalizes $\rho^{(12)}$ (similar properties hold for the density matrix $\rho^{(23)}$.) Let us call m the number of states kept per block in a DMRG computation. This number typically varies between 10 and 1000 depending on the computer resources.

The DMRG algorithm consists in choosing the m most probable states α , i.e. the states with higher value of w_{α} (let us sort them as $(w_1 \geq w_2 \geq w_3 \geq \dots \geq w_{m_1 m_2})$) This guarantees

the best possible representation of the target state ψ for every given value of m . Moreover the sum $P(m) = \sum_{\alpha=1}^m w_\alpha$ of the probabilities of the m states kept give a reasonable measure of the truncation error (recall that $\text{tr}\rho^{(12)} = \sum_\alpha w_\alpha = 1$ and hence $P(m) \leq 1$).

In many of the 1d models studied with the DMRG it turns out that the probability w_α is concentrated in a few states and that it decays exponentially fast. This implies that with small values of m one can achieve a great accuracy in representing the target state. This is certainly the case for systems with a finite correlation length [7,13,14]. For systems with an infinite correlation length one has to study finite systems and adjust the number of states kept m to the correlation length due the finite size [15].

THE DMRG VERSUS QUANTUM GROUP THEORY AND CFT

There are certain formal analogies between the DMRG and the theory of quantum groups and Conformal Field Theories (CFT) which we shall review below. First of all the DMRG truncation of states in $(B_1 B_2)'$ has strong similarities with the truncated tensor product of irreps of a q -Group where q is a root of unit [16]. Let us choose for example the quantum group $SU(2)_q$ which is a q -deformation of the rotation group $SU(2)$. For generic values of q the representation theory of $SU(2)_q$ is similar to that of $SU(2)$, i.e. every irrep corresponds to an integer or half integer spin $j = 0, 1/2, \dots$ and the tensor product of irreps satisfies the standard Clebsch-Gordan decomposition. However, if q is a root of unit, $q = e^{2\pi i/(k+2)}$ then there is only a finite number of regular irreps corresponding to the spins $j = 0, 1/2, \dots, k/2$. The tensor product of these irreps is a truncated version of the classical CG decomposition,

$$(V_{j_1} \otimes V_{j_2})' = \bigoplus_{j=|j_1-j_2|}^{\min(j_1+j_2, k-j_1-j_2)} V_j \quad (18)$$

V_j denotes the vector space of dimension $2j+1$ associated to the irrep with spin j . It is interesting to observe that the truncated tensor product (18) satisfies the associativity condition (8), namely [16],

$$((V_1 \otimes V_2)' \otimes V_3)' = (V_1 \otimes (V_2 \otimes V_3)')' \quad (19)$$

This eq. is a consequence of the co-associativity of the comultiplication of the quantum group $SU(2)_q$. In more physical terms, eq.(19) follows from the non trivial addition rule of angular momenta in $SU(2)_q$. The regular irreps have positive q -dimension which is defined as [16],

$$d_j \equiv [2j + 1]_q \equiv \frac{q^{(2j+1)/2} - q^{-(2j+1)/2}}{q^{1/2} - q^{-1/2}} \quad (20)$$

The q -dimension of an irrep plays a role similar to the eigenvalues w_α of the density matrix in the sense that irreps with zero q -dimension are thrown away in the tensor product just like in the DMRG truncation. Based on this analogy we conjecture that a q -group invariant Hamiltonian, like the XXZ open chain with q a root of unit, when studied with DMRG methods will yield a density matrix with vanishing eigenvalues corresponding to non regular irreps. The DMRG truncation of these states have to agree with the q -group truncation of the non regular states [6].

Quantum groups with q a root of unit are intimately related to rational CFT's (RCFT). Indeed in a RCFT there is a finite number of primary fields $\phi_a (a = 1, \dots, M)$, which are in one-to-one correspondence with the regular irreps of the associated q -group [16]. Hence from the previous relation between the DMRG and q -groups we may expect a relationship between RCFT's and the DMRG. More generally in a CFT there are null states in the Verma modules of the primary fields, whose norm is zero. As shown by Belavin, Polyakov and Zamolodchikov (BPZ) the decoupling of null vectors leads to a set of partial differential equations for the conformal blocks of the theory, in terms of which one can construct all the correlators of the theory [17]. It is tempting to suggest that the BPZ decoupling of null vectors is the field theoretical version of the DMRG truncation. On the other hand the analogue of the tensor product decomposition is given by the fusion rules of the primary fields,

$$\phi_a \otimes \phi_b = N_{a,b}^c \phi_c \quad (21)$$

where $N_{a,b}^c$ is an integer which counts how many times the primary field ϕ_c appears into the Operator Product Expansion (OPE) of ϕ_a and ϕ_b . The associativity of the OPE, i.e.

$$((\phi_a \otimes \phi_b) \otimes \phi_c) = (\phi_a \otimes (\phi_b \otimes \phi_c)) \quad (22)$$

implies a non trivial eq. for the fusion coefficients $N_{a,b}^c$ namely,

$$\sum_d N_{a,b}^d N_{d,c}^f = \sum_d N_{a,d}^f N_{b,c}^d \quad (23)$$

An example of RCFT is given by the $SU(2)_k$ WZW model with level k [18]. The primary fields ϕ_j are labelled by the spin $j = 0, 1/2, \dots, k/2$, while the fusion rules are given by the eq.(18) with the translation $V_j \rightarrow \phi_j$. Indeed, as shown in reference [16], there is a one-to-one correspondence between the $SU(2)_k$ WZW model and the quantum group $SU(2)_q$.

Another aspect of the relation between CFT, Integrable Systems and the DMRG concerns the explanation of the exponential decay of the eigenvalues of the density matrix. An approach to study this connection is through the relation between the DMRG density matrix and the Corner Transfer Matrix (CTM) of Baxter first pointed out by Nishino [19] in his application of the DMRG to classical statistical mechanical models in 2D. As shown by Baxter [20] in an integrable system the eigenvalues of the CTM have a very simple structure, i.e. they go as a^n with n an integer. One can recognize here the exponential decay of the eigenvalues of the density matrix [21], [22].

There are still many aspects to clarify in the relation between the DMRG and CFT and more generally integrable systems. This could be a fruitful subject in the near future.

THE DMRG AND THE MATRIX PRODUCT ANSATZS

The DMRG is a variational method which generates an ansatz for the GS state and the excited states. This implies in particular that the DMRG ground state (GS) energy is an upper bound of the exact GS energy. The variational ansatz generated by the DMRG is of the matrix product (MP) type. This fact was shown by Östlund and Rommer in the thermodynamic limit of the DMRG in the case of the spin 1 chain [7]. These authors proposed that one could get very good results for the GS energy and spin gap by using a

MP ansatz which corresponds to a small value of m in the DMRG. The excitations could also be constructed as Bloch waves on the MPM state.

To understand why the DMRG gives rise to a MP state we return to eq.(13). If $|i_1\rangle, |i_2\rangle$ and $|\alpha\rangle$ denote basis of the Hilbert spaces associated to B_1, B_2 and $(B_1 B_2)'$, then the relation between these basis is

$$|\alpha\rangle = \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} U_{i_1 i_2, \alpha} |i_1\rangle |i_2\rangle, \quad (\alpha = 1, \dots, m) \quad (24)$$

Since B_2 is usually a lattice site \bullet we shall write eq.(24) in the following form,

$$|\alpha\rangle_N = \sum_{\beta, s} A_{\alpha, \beta}^N[s] |\beta\rangle_{N-1} |s_N\rangle \quad (25)$$

where $|s_N\rangle$ denotes the local state associated to the site located at the N^{th} position of the chain, while $|\alpha\rangle_N$ and $|\beta\rangle_{N-1}$ are the states kept for the blocks of lengths N and $N - 1$ respectively. $A_{\alpha, \beta}^N[s]$ is a matrix $m \times m$ for each value of s .

Iterating (25) until reaching the boundary of the chain one gets,

$$|\alpha_N, \alpha_0\rangle_N = \left(A^N[s_N] A^{N-1}[s_{N-1}] \dots A^1[s_1] \right)_{\alpha_N, \alpha_0} |s_1\rangle \dots |s_{N-1}\rangle |s_N\rangle \quad (26)$$

where the matrix multiplication of the $A^n[s_n]$ matrices is implicit. $|\alpha_N, \alpha_0\rangle$ is a collection of states of an open chain with N sites labelled by the pair (α_N, α_0) . For a closed chain with periodic boundary conditions the ansatz becomes

$$|\psi\rangle_N = \text{Tr}(A^N[s_N] A^{N-1}[s_{N-1}] \dots A^1[s_1]) |s_1\rangle \dots |s_{N-1}\rangle |s_N\rangle \quad (27)$$

A further simplification of (27) is to assume that all the matrices $A^n[s_n]$ are independent on n , i.e. $A^n[s_n] = A[s_n]$ (for all n). This assumption can be justified in the thermodynamic limit of the DMRG where it reaches a fixed point [7]. However for finite dimensional systems and specially for open BC's there will be a non trivial dependence of $A^n[s_n]$ on n . In this sense the DMRG gives a non homogenous MP ansatz.

In eq.(25) we may want that the states $|\alpha\rangle$ form an orthonormal set of states given that both $|\beta\rangle$ and $|s\rangle$ are orthonormal sets. This implies the following eq. on $A^N[s]$

$$\sum_{\beta,s} (A_{\alpha,\beta}^N[s])^* A_{\alpha',\beta}^N[s] = \delta_{\alpha,\alpha'} \quad (28)$$

which is nothing else than the eq. (14). This eq. expresses the fact that A^N relates orthonormal basis. But recall that it is not simply a change of basis because we are truncating states, i.e. $m < m_1 m_2$.

Given the MP ansatzs (26) and (27) for open and closed chains respectively we can use a standard variational method to find the amplitudes $A[s]$ which minimize the energy of the ansatz. In references [13,14] it was shown that when N is large these minimization procedure is similar to the one of the DMRG and that in fact there is a hidden density matrix even though the algorithm did not try to follow the DMRG method. One way to see this is if one define the following transfer matrix,

$$T_{\alpha\alpha',\beta\beta'}^N = \sum_s (A_{\alpha,\beta}^N[s])^* A_{\alpha',\beta'}^N[s] \quad (29)$$

T^N is a $m^2 \times m^2$ matrix which serves to relate matrix elements of operators between states with lengths N and $N-1$, for example

$${}_N \langle \alpha | \mathcal{O} | \alpha' \rangle_N = \sum_{\beta,\beta'} T_{\alpha\alpha',\beta\beta'}^N {}_{N-1} \langle \beta | \mathcal{O} | \beta' \rangle_{N-1} \quad (30)$$

We are assuming in (30) that the operator \mathcal{O} does not act on the N^{th} site. The normalization condition (28) implies that T has a right eigenvector with eigenvalue 1 given by $\delta_{\alpha,\alpha'}$, namely

$$\sum_{\beta\beta'} T_{\alpha\alpha',\beta\beta'}^N \delta_{\beta,\beta'} = \delta_{\alpha,\alpha'} \quad (31)$$

It then follows that T has a left eigenvector with eigenvalue 1, i.e.

$$\sum_{\alpha\alpha'} \rho_{\alpha\alpha'}^N T_{\alpha\alpha',\beta\beta'}^N = \rho_{\beta,\beta'}^N \quad (32)$$

In references [13,14] it was shown that $\rho_{\alpha,\alpha'}^N$ can be identified with a density matrix and that one is really minimizing the expectation value of the Hamiltonian in the following mixed state,

$$\rho^N = \sum_{\alpha\alpha'} \rho_{\alpha\alpha'}^N |\alpha\rangle_N {}_N \langle \alpha'| \quad (33)$$

From this point of view the collection of states $|\alpha\rangle_N$ of the MP method can be interpreted as the most probable ones that contribute to the GS wave function of a system with $N + 1 + N = 2N + 1$ sites.

The important conclusion to be learn from the previous considerations is that the MP ansatz leads in a natural way to the DMRG algorithm. This may be interesting regarding further generalizations of the DMRG to higher dimensions.

MATRIX PRODUCTS ANSATZS IN 2D

The DMRG algorithm can be generalized to ladders (i.e. collections of a finite number of chains), and large clusters. This has been done obtaining remarkable results which are difficult to obtain with other algorithms [23]. However it has also been shown that the efficiency of the DMRG diminish with the width of the system [24]. The DMRG algorithm appears to be essentially one dimensional in the sense that the RG steps follow a linear pattern no matter whether the system is 1D or higher dimensional.

Of course any higher dimensional system can be converted into a 1D system by allowing non local interactions. However locality seems to be the key of the great performance of the DMRG in 1D. Hence a truly higher dimensional version of the DMRG should try to keep locality as a guideline. That this is in principle possible is suggested by reference [25] where a DMRG algorithm is given for a Bethe lattice, whose dimensionality is actually infinite. Also recently, Niggemann et al. [26] have constructed a two-dimensional tensor product to describe the ground state of a 2D quantum system. Similarly, Nishino and Okunishi have proposed a Density Matrix algorithm for 3D classical statistical mechanics models [27].

Another approach is suggested by the equivalence of the Matrix Product approach and the DMRG for 1d or quasi-1d systems. We do not know at the moment what is the formulation of the DMRG in 2D but we do know that in 2D there are MP states which were first constructed by Affleck, Kennedy, Lieb and Tasaki (AKLT) [28]. These states are valence bond solid states where one connects local spins through local bonds.

When trying to generalize the 1D MP states to 2D we find that there are two possible types of MP states which can be conveniently named as vertex-MP and face-MP states, using standard Statistical Mechanics terminology [20].

Vertex-Matrix Product ansatzs

A vertex model in 2D Statistical Mechanics (SM) is a model defined on a square lattice and such that the lattice variables i, j, \dots live on the links while the interaction takes place on the vertices [20]. The Boltzmann weight thus depends on 4 variables W_{ijkl} and the whole partition function is obtained by multiplying the Boltzmann weights of all vertices and then summing over the values of the lattice variables. The 6 vertex and 8 vertex models are the canonical examples of these types of models which have been shown to be integrable.

Motivated by these vertex models we shall define a vertex-MP state in terms of a set of amplitudes

$$A_{\alpha,\beta}^{\gamma,\delta}[s], \quad (\alpha, \beta, \dots = 1, \dots, m; \quad s = 1, \dots, m_s) \quad (34)$$

where the labels $\alpha, \beta, \gamma, \delta$ are associated with the links of the square lattice while s labels the quantum state, e.g. spin, associated to the vertex where the 4-links $\alpha, \beta, \gamma, \delta$ meet. $A_{\alpha,\beta}^{\gamma,\delta}[s]$ is a sort of Boltzmann weight of a vertex model. The vertex-MP wave function $\psi(s_1, s_2, \dots, s_N)$ can be obtained by multiplying all the Boltzmann weights $A_{\alpha_i, \beta_i}^{\gamma_i, \delta_i}[s_i]$ and contracting and summing over the links variables according to the same pattern of a vertex model in Statistical Mechanics [20]. Hence the value of the wave function $\psi(s_1, s_2, \dots, s_N)$ is given by the partition function of a vertex model where the Boltzmann weights depend on the value of the local states s_i . This construction for the square lattice is equivalent to the so called “vertex-state representation” of Niggemann et al. for the hexagonal lattice [?].

This construction resembles the one proposed by Laughlin concerning the Fractional Quantum Hall effect (FQHE) [30]. More explicitly, Laughlin proposed in [30] a variational wave function $\psi_m(z_1, \dots, z_N)$ for the ground state of the N electrons in the lowest Landau

level of a FQHE with filling factor $\nu = 1/m$. The norm $|\psi_m|^2$ of the Laughlin wave function can be interpreted as the Boltzman weight of a classical one component plasma constituted by N negative charges of magnitude m in a uniform background of positive charges. The charge neutrality of the plasma guarantees its stability.

In our case we also have an associated Statistical Mechanical model given by the Boltzman weights of a vertex model. If we compute the norm of the wave function $\psi(s_1, \dots, s_N)$, we can perform the summation over the “spin” indices s , in which case the the norm $\langle \psi | \psi \rangle$ of the vertex-MP state is given by the partition function of another vertex model whose Boltzmann weights are defined as,

$$R_{\alpha\alpha',\beta\beta'}^{\gamma\gamma',\delta\delta'} = \sum_s A_{\alpha,\beta}^{\gamma,\delta}[s] A_{\alpha',\beta'}^{\gamma',\delta'}[s] \quad (35)$$

This R matrix is the 2D version of the T matrix defined in (29). The computation of the norm of $|\psi\rangle$ can be in general a difficult task. However, if the model defined by the weights (35) turns out to be integrable, then we could find the exact norm in the thermodynamic limit.

The face-MP models can be defined in a similar manner by a set of variational parameters as in (34) where now the variables α, \dots are now associated to the vertices of the squares while the quantum variable s is associated to the face whose vertices are $\alpha, \beta, \gamma, \delta$. This is similar to the face or Interaction Round a Face models (IRF) in Statistical Mechanics [20].

Hence in 2D there are two generic ways to produce MP ansatzs which are in fact the straightforward generalization of the 1D MP ansatzs. These two generalizations suggest to use some well know models as the 6-vertex model to test some of the ideas presented above.

In summary, we have tried to show in this contribution some interesting connections among seemingly unrelated methods in condensed matter and field theory. Much remains to be done along this direction.

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